Algebraic Topology (Fall 2018): Final Exam

1. Show that the degree of a map $f\colon T^n \to T^n$ of the n dimensional torus is the determinant of $f^*\colon H^1(T^n;\mathbb{Z}) \to H^1(T^n;\mathbb{Z})$.

2. For a map $f\colon M \to N$ between connected closed orientable n-dimensional manifolds with fundamental classes $[M]$ and $[N]$, the degree of f, denoted by $d(f)$, is defined to be the integer d such that $f_*([M]) = d[N]$. (See Exercise #7 and #8 in Section 3.3 (Page 259) of Hatcher's book.)

   Using Poincaré duality and the naturality property of cap products, show that a map $f\colon M \to N$ of degree 1 between closed orientable n-dimensional manifolds induces split surjections $f_*\colon H_i(M;R) \to H_i(N;R)$ for all i.

3. For a map $f\colon M \to N$ between connected closed n-dimensional manifolds with $\mathbb{Z}/2\mathbb{Z}$-fundamental classes $[M]$ and $[N]$. The degree of f modulo 2, denoted by $d_2(f)$, is defined to be integer $k \in \mathbb{Z}/2\mathbb{Z}$ such that $f_*[M] = k[N]$. Show that if this degree is non-zero, then f is surjective. Moreover, if the manifolds are oriented, then $d(f) \equiv d_2(f) \pmod 2$, where $d(f)$ is the usual degree of f.

4. Show that if M is a compact R-orientable n-manifold, then the boundary map

$$H_n(M,\partial M;R) \to H_{n-1}(\partial M;R)$$

   sends a fundamental class for $(M,\partial M)$ to a fundamental class for ∂M.

5. If W is a compact manifold with boundary $\partial W = M$. Show that the Euler characteristic $\chi(M)$ of M is even. (Hint: apply Mayer-Vietoris sequence to the double of W. Here the double of W is the closed manifold obtained by gluing two copies of W along the common boundary $\partial W = M$).

6. Let M be a closed connected oriented n-dimensional manifold with fundamental class $[M]$. Let $H^*(M;\mathbb{R})$ be the cohomology groups of M with coefficients in the real numbers $\mathbb{R}$. In the question, we will work with coefficients $\mathbb{R}$ exclusively, and will omit $\mathbb{R}$ from our notation. We have seen that the cup product induces the following bilinear map

$$\Phi\colon H^i(M) \times H^{n-i}(M) \to \mathbb{R}$$

   by setting $\Phi(\alpha,\beta) = (\alpha \cup \beta)[M]$.

   (a) If $n = 4k$, then in the middle dimension we have

$$\varphi\colon H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}.$$

   Show that φ is a symmetric bilinear map, that is $\varphi(\alpha,\beta) = \varphi(\beta,\alpha)$. Moreover, φ is nonsingular. Similarly, show that if $n = 4k+2$,

$$\varphi\colon H^{2k+1}(M) \times H^{2k+1}(M) \to \mathbb{R}$$

   is a nonsingular skew-symmetric bilinear map.

A symmetric bilinear map $\varphi\colon V \times V \to \mathbb{R}$ on a real vector space V is just a symmetric matrix A, after fixing a basis of V. Recall that every symmetric matrix is congruent to a diagonal matrix with 1's, -1's and 0's along the diagonal. That is, there exists another basis of V such that A is diagonal with respect to that basis. Moreover, the number of 1's, the number of -1's, and the number of 0's are independent of the choice of basis. This is usually referred to as Sylvester's law of inertia.

Denote the number of 1's by r and the number of -1's by s. Then the *signature* of φ is defined to be the integer $r - s$. Now if $\dim M = 4k$, we define the *signature*, denoted by $\tau(M)$, of M to be the signature of the symmetric bilinear map

$$\varphi\colon H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}.$$

If $\dim M$ is not a multiple of 4, then we define its signature to be zero.

It is clear that the notion of signature also makes sense for closed oriented manifolds with finitely many connected components. Suppose M is closed oriented manifold with connected components M_i, for $1 \le i \le \ell$. Then we define

$$\tau(M) = \sum_{i=1}^{\ell} \tau(M_i).$$

(b) Let M and N be closed connected oriented manifolds with fundamental class $[M]$ and $[N]$ respectively. Suppose $\dim M = m$ and $\dim N = n$. Equip $M \times N$ with the product orientation, that is, the fundamental class of $M \times N$ is $[M] \otimes [N] \in H_{m+n}(M \times N) \cong H_m(M) \otimes H_n(N)$. Show that $\tau(M \times N) = \tau(M) \cdot \tau(N)$.
Hint: use Künneth formula for cohomology.

(c) Suppose W is compact connected oriented $(4k + 1)$-dimensional manifold with boundary $\partial W = M$. Show that $\tau(M) = 0$ by the filling the details of the following steps.

(i) Let $\iota\colon M \to W$ be the inclusion. Suppose V is the image of the map

$$\iota^*\colon H^{2k}(W) \to H^{2k}(M).$$

Show that the symmetric bilinear map $\varphi\colon H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$ is identically zero on V. (Hint: use the fact that the connecting map

$$H_{4k+1}(W, \partial W) \to H^{4k}(M)$$

maps the fundamental class $[W] \in H_{4k+1}(W, \partial W)$ to the fundamental class $[M] \in H_{4k}(M)$).

(ii) Use Poincaré duality to show that $\dim V$ is equal to the dimension of the kernel of the map

$$\iota_*\colon H_{2k}(M) \to H_{2k}(W).$$

(iii) Use universal coefficient theorem to show that $\dim V$ is equal to the dimension of the image of the map

$$\iota_*\colon H_{2k}(M) \to H_{2k}(W).$$

(iv) Combining (ii) and (iii), show that

$$\dim V = \frac{\dim H^{2k}(M)}{2}.$$

Now recall the following fact. Suppose φ is a symmetric nonsingular bilinear form on a real vector space X of dimension $2n$. If there exists a subspace $V \subset X$ of dimension n such that φ is identically zero on V, then the signature of φ is zero. You do not need to prove this fact. Use it to show that $\tau(M) = 0$.